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Masters Thesis

Summation Chains of Sequences
Part 1: Introduction, Generation, and Key Definitions

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Abstract
(Editor's note:) This paper represents the first installment of a masters thesis by Jonathan Johnson. This work introduces the notion of summation chains of sequences. It examines the sequence of sequences generated by partial sums and differences of terms in each level of the chain, looks at chains generated by functions, then introduces a formal definition and key formulae in the analysis of such chains.

Keywords
sequences

Contents

1. Introduction

Given a complex-valued sequence $(a_n)_{n=1}^{\infty}$, the *sequence of partial sums* of (a_n) is given by the sequence $(a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, \sum_{i=1}^{n} a_i, \ldots)$. The *sequence of differences* of (a_n) is given by the sequence $(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}, \ldots)$. The processes of finding the sequence of partial sums and finding the sequence of differences of a sequence are inverses of each other so every sequence is the sequence of differences of its sequence of partial sums and the sequence of partial sums of its sequence of differences. Every sequence has a unique sequence of partial sums and a unique sequence of differences so it is always possible to find the sequence of partial sums of the sequence of partial sums and repeat the process ad infinitum. Similarly, we can find the sequence of differences of the sequence of differences and repeat ad infinitum. The result is a doubly infinite sequence or "chain" of sequences where each sequence is the sequence of partial sums of the previous sequence and the sequence of differences of the following sequence.

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Example 1.1. Let $a^{(0)}$ be the sequence defined by $a_n^{(0)} = (-1)^{n+1}$, for all $n \in \mathbb{N}$. For all integers $m > 0$, let $a^{(m)}$ be the sequence of partial sums of $a^{(m-1)}$, and for all integers $m < 0$, let $a^{(m)}$ be the sequence of differences of $a^{(m+1)}$.

Every sequence can be used to create a unique chain of sequences. This paper studies properties of these chains and explores the relationship between sequences and the chains they create. In particular, the following questions are investigated:

- *How can the sequences in a summation chain be computed quickly?* Clearly, every sequence in a chain of sequences can be computed by repeatedly finding sequences of partial sums of sequences of partial sums or finding sequences of differences of sequences of differences. It is useful, however, to be able compute any sequence in a chain given the starting sequence, $a^{(0)}$, without having to compute all the sequences in between. Methods for computing chains are discussed in Section 3.
- *When do two given sequences appear in the same summation chain?* When two sequences appear in the same chain, one sequence can be obtained by repeatedly finding the sequences of partial sums of sequences of partial sums of the other sequence. This process could take a long time, and it is not able to determine if two sequences do not appear in the same chain. (*Editor's note:) The next installment presents a process to determine with certainty whether or not two sequences are in the same chain.*
- *How much information is needed to define a summation chain?* Once a chain has been computed, it appears as an array of entries. Starting with a blank array, if some numbers are added to a blank array, can they be used to define the remaining entries uniquely? (*Editor's note:) Later installments explore how much information in an array of numbers is needed to determine a chain.*
- *How are the convergent behaviors of sequences in a summation chain related?* Can every sequence in a chain diverge? Can every sequence in a chain converge? (*Editor's note:) The final installment investigates the nature of the limits of sequences in a chain.*

2. Chains Generated by Functions

The summation chains we have considered are doubly infinite sequences (of sequences) in which a fixed rule determines the chain of sequences recursively in both directions. The recursive rule in the backward direction is the inverse of the rule in the forward direction. Unlike single infinite sequences, in a chain there is no starting point to begin the recursive pattern. A sequence must be chosen to "start" the chain, and could be thought of as a "seed" of the chain. Starting with a given sequence, the next sequence in the chain can be obtained by applying a rule, and the previous sequence can be obtained by applying the rule's inverse. However, this same pattern of sequences could be generated starting with any other sequence in the chain, together with the same generating rule. In order for this process to be well-defined, the rule used must have a well-defined inverse. The following definition captures abstractly what chains are. It also generalizes chains of sequences to chains of elements of more general sets.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$a^{(2)}$	1	1	2	2	3	3	4	4	$\cdots$
$a^{(1)}$	1	0	1	0	1	0	1	0	$\cdots$
$a^{(0)}$	1	−1	1	−1	1	−1	1	−1	$\cdots$
$a^{(-1)}$	1	−2	2	−2	2	−2	2	−2	$\cdots$
$a^{(-2)}$	1	−3	4	−4	4	−4	4	−4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Figure 1.1. Example 1.1

Definition 2.1. Let $f : X \to X$ be a bijective function on a set X. We will call f a *function rule*. An f*-chain* is any function $c : \mathbb{Z} \to X$ such that $c(m) = f(c(m-1))$ for all $m \in \mathbb{Z}$. $c(0) \in X$ is called the *seed* of c, and c is *generated by* f with seed $c(0)$. $c(m)$ is called the mth *term* of the chain.

Each term of a chain is given uniquely as $c(m) = f^m(c(0))$.

Definition 2.2. Let f be a function rule on X. Define $\mathcal{C}_f$ to be the set of all f-chains. The f-chain *seed mapping* is the function $\Phi_f : X \to \mathcal{C}_f$ where $\Phi_f(x)$ is the chain generated by the seed x.

Remark 2.1. Φ_f is bijective.

Example 2.1. Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ where $f(n) = 2n$.

This function has five chains generated by 0, 1, 2, 3, and 4. (Figure 2.1) Notice that the chains generated by 1, 2, 3, and 4 contain the same pattern differing only by a shift in starting point.

$$\begin{array}{rcccccccccc} m: & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ c(m): & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

Chain Seeded by 0

$$\begin{array}{rcccccccccc} m: & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ c(m): & \cdots & 2 & 4 & 3 & 1 & 2 & 4 & 3 & \cdots \end{array}$$

Chain Seeded by 1

$$\begin{array}{rcccccccccc} m: & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\ c(m): & \cdots & 1 & 2 & 4 & 3 & 1 & 2 & 4 & \cdots \end{array}$$

Chain Seeded by 3

Figure 2.1. Chains Generated by f

Definition 2.3. Let f be a function rule on X. Let x_1 and x_2 be in X. x_1 is f*-chain related* to x_2, denoted $x_1 \overset{f}{\sim} x_2$, if x_1 is in the image of the function chain $\Phi_f(x_2)$.

Proposition 2.1. *An f-chain relation is an equivalence relation.*

Proof. Let f be a function rule on set X.

Let x, y, and z be arbitrarily in X. $(\Phi_f(x))(0) = x$ so $x \overset{f}{\sim} x$.

If $x \overset{f}{\sim} y$, then there exists $m \in \mathbb{Z}$ such that $x = f^m(y)$ so $y = f^{-m}(x) = (\Phi_f(x))(-m)$ so $y \overset{f}{\sim} x$.

If $x \overset{f}{\sim} y$ and $y \overset{f}{\sim} z$, then there exists $m, n \in \mathbb{Z}$ such that $x = f^m(y)$ and $y = f^n(z)$.

Therefore, $x = f^m(f^n(z)) = f^{m+n}(z) = (\Phi_f(z))(m+n)$ so $x \overset{f}{\sim} z$. □

This relation is useful to find relationships between elements in the set X induced by f. In the Example 2.1 above, 1, 2, 3, and 4 are all f-chain related while 0 is related only to itself.

This section ends with a few notable properties of chains that are useful, in particular when dealing with summation chains.

Proposition 2.2. *Let c_1 and c_2 be f-chains. If for any $m \in \mathbb{Z}$, $c_1(m) = c_2(m)$, then $c_1 = c_2$.*

Proof. If $c_1(m) = c_2(m)$, then $c_1(0) = f^{-m}(c_1(m)) = f^{-m}(c_2(m)) = c_2(0)$.
Therefore, $c_1 = \Phi_f(c_1(0)) = \Phi_f(c_2(0)) = c_2$. □

Proposition 2.3. *Let X be a set with rule f. Let c be an f-chain with co-domain X, and let function $d : \mathbb{Z} \to X$ be such that for some $k \in \mathbb{Z}$, $d(m) = c(m+k)$ for all $m \in \mathbb{Z}$, then d is an f-chain. Also, for every pair $x \in im(c)$ and $y \in Im(d)$, $x \overset{f}{\sim} y$.*

Proof. $\forall m \in \mathbb{Z}$, $d(m) = c(m-k) = f(c(m-k-1)) = f(d(m-1))$

Let $x \in \mathrm{Im}(c)$ and $y \in \mathrm{Im}(d)$. $x \overset{f}{\sim} d(0) \overset{f}{\sim} y$ □

A more detained analysis of sequences generated by recursive funtions in Brin and Stuck (2). The remainder of this paper will focus on summation chains of sequences.

3. Summation Chains of Sequence

3.1 Key Definition and Formulas

Notation Let M be a set, and let X be the set of sequences of elements in M. Given a function rule f on X, and an f-chain, c, in order to keep notation clean, new notation is defined as follows. $c(m,n) :=$ the nth term of the sequence $c(m)$.

Using this notation we give a clean definition of a summation chain of sequences.

Definition 3.1. Let M be a $\mathbb{Z}$-Module. A *summation chain*, denoted Σ-chain, of M is a function $c : \mathbb{Z} \times \mathbb{N} \to M$ where $c(m,n) = \sum_{k=1}^{n} c(m-1,k)$, for all $m \in \mathbb{Z}$ and for all $n \in \mathbb{N}$.

$\Phi_{\Sigma[M]}$ is the *seed mapping* for summation chains in M. The notation Φ_Σ is used when the $\mathbb{Z}$-Module, M, is clear.

$\mathcal{C}_{\Sigma[M]}$ denotes the set of all M-valued summation chains.

Remark 3.1. Summation chains are generated by the function rule $T_\Sigma : M^\infty \to M^\infty$ where $T_\Sigma(x_1, x_2, x_3, \ldots) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots)$.

Example 3.1. Let $t : \mathbb{Z} \times \mathbb{N} \to M$ be the constant function $t(m,n) = 0$.

$\sum_{k=1}^{n} t(m-1,k) = \sum_{k=1}^{n} 0 = 0 = t(m,n)$ so t is a Σ-chain. This chain is the trivial chain and is generated with the sequence of all zeros.

How can the sequences in a summation chain be computed quickly? The next two results provide ways to compute entries of a summation chain. Proposition 3.1 provides a way to compute the entries in a chain using the other entries close to it.

Proposition 3.1. *Let M be a $\mathbb{Z}$-Module. Let $c : \mathbb{Z} \times \mathbb{N} \to M$ be a function. c is a Σ-chain in M if and only if for all $m, m' \in \mathbb{Z}$ and $n \in \mathbb{N}$, the following equalities hold.*

(i) $c(m,1) = c(m',1)$,

(ii) If $n \geq 2$, then $c(m,n) = c(m,n-1) + c(m-1,n) = c(m+1,n) - c(m+1,n-1)$.

Proof. Assume c is a Σ-chain in M. The first term of a sequence and its sequence of partial sums is always the same so by induction the (i) holds. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$.

$$\begin{aligned} c(m,n) &= \sum_{k=1}^{n} c(m-1,k) \text{ by Definition 3.1} \\ &= \sum_{k=1}^{n-1} c(m-1,k) + c(m-1,n) \\ &= c(m,n-1) + c(m-1,n) \end{aligned} \tag{1}$$

By equation (1),

$$c(m+1,n) = c(m+1,n-1) + c(m,n) \Rightarrow c(m,n) = c(m+1,n) - c(m+1,n-1)$$

Assume both (i) and (ii). $c(m,1) = \sum_{k=1}^{1} c(m-1,k)$ by (i).
If $c(m,n) = \sum_{k=1}^{n} c(m-1,k)$, then by (ii),

$$\begin{aligned} c(m,n+1) &= c(m,n) + c(m-1,n+1) \\ &= \sum_{k=1}^{n} c(m-1,k) + c(m-1,n+1) \\ &= \sum_{k=1}^{n+1} c(m-1,k) \end{aligned} \tag{2}$$

So by induction, c is a Σ-chain. □

The next theorem provides a way to calculate any sequence in a summation chain directly given any other sequence in the chain without having to compute any of the sequences in between.

Lemma 3.2. *Let natural numbers, n and k, where $k < n$.*

$$\binom{n}{k} = \binom{n-1}{k-1}\binom{n-1}{k} \tag{3}$$

Proof. A proof this can be found in Cameron [3]. □

Theorem 3.3. *Let c be a Σ-chain in $\mathbb{Z}$-Module M. Let $m_1, m_2 \in \mathbb{Z}$ such that $m_1 > m_2$, and let $n \in \mathbb{N}$. Define $m = m_1 - m_2$.*

$$c(m_1,n) = \sum_{k=1}^{n} \binom{m+n-k-1}{m-1} c(m_2,k) \tag{4}$$

If $n \geq m+1$,

$$c(m_2,n) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{m-k} c(m_1,n-m+k) \tag{5}$$

If $n \leq m+1$,

$$c(m_2,n) = \sum_{k=1}^{n} (-1)^{n-k} \binom{m}{n-k} c(m_1,k) \tag{6}$$

Remark 3.2. When $n = m+1$, (5) and (6) are the same.

Proof of this theorem is given in Section 3.2.

Example 3.2. Let $c = \Phi_{\Sigma[\mathbb{R}]}((2^{k-1}))$. The following is a computation of the sequences $c(-5,n)$ and $c(-6,n)$.

$c(-5,1) = c(0,1) = 1.$
$c(-5,2) = c(0,2) - 5c(0,1) = 2 - 5(1) = -3.$
$c(-5,3) = c(0,3) - 5c(0,2) + 10c(0,1) = 4 - 5(2) + 10(1) = 4.$
$c(-5,4) = c(0,4) - 5c(0,3) + 10c(0,2) - 10c(0,1) = 8 - 5(4) + 10((2) - 10(1) = -2.$
$c(-5,5) = c(0,5) - 5c(0,4) + 10c(0,3) - 10c(0,2) + 5c(0,1) = 16 - 5(8) + 10(4) - 10((2) + 5(1) = 1.$

When $n \geq 6$,

$$\begin{aligned} c(-5,n) &= \sum_{i=0}^{4} \binom{4}{i} c(0,n-i) \\ &= 2^{n-1} - 5 \cdot 2^{n-2} + 10 \cdot 2^{n-3} - 10 \cdot 2^{n-4} + 5 \cdot 2^{n-5} - 2^{n-6} \\ &= 2^{n-6}(32 - 5 \cdot 16 + 10 \cdot 6 - 10 \cdot 4 + 5 \cdot 2 - 1) \\ &= 2^{n-6}(1) \\ &= 2^{n-6} \end{aligned} \tag{7}$$

$c(-6,1) = c(-5,1) = 1$
$c(-6,2) = c(-5,2) - c(-5,1) = -3 - 1 = -4$
$c(-6,3) = c(-5,3) - c(-5,2) = 4 - (-3) = 7$
$c(-6,4) = c(-5,4) - c(-5,3) = -2 - 4 = -6$
$c(-6,5) = c(-5,5) - c(-5,4) = 1 - (-2) = 3$
$c(-6,6) = c(-5,6) - c(-5,5) = 1 - 1 = 0$
When $n \geq 7$, $c(-6,n) = c(-5,n) - c(-5,n-1) = 2^{n-6} - 2^{n-7} = 2^{n-7}$

Example 3.3. Let $a_1 = 1$, $a_2 = -1$, and $a_k = 0$ when $k \geq 3$. Let $c = \Phi_{\Sigma[\mathbb{R}]}((a_k))$. The following is a computation of the sequence $c(4,n)$.

$$c(4,1) = c(0,1) = a_1 = 1$$

When $n \geq 2$,

$$\begin{aligned}
c(4,n) &= \sum_{k=1}^{n} \binom{3+n-k}{3} c(0,k) \\
&= \binom{n+2}{3} - \binom{n+1}{3} \\
&= \frac{n(n+1)(n+2)}{6} - \frac{(n-1)n(n+1)}{6} \\
&= \frac{n(n+1)[(n+2)-(n-1)]}{6} \\
&= \frac{n(n+1)}{2}
\end{aligned} \tag{8}$$

3.2 Proof of Theorem 3.3

Proof. The following proof uses induction on n nested inside an induction argument on m. First, assume that $m = m_1 - m_2 = 1$.

$$\begin{aligned}
c(m_1,n) &= \sum_{k=1}^{n} c(m_1-1,k) \text{ by Definition 3.1} \\
&= \sum_{k=1}^{n} \binom{m+n-k-1}{0} c(m_2,k) \\
&= \sum_{k=1}^{n} \binom{m+n-k-1}{m-1} c(m_2,k)
\end{aligned} \tag{9}$$

If $n \geq m+1$ then

$$\begin{aligned}
c(m_2,n) &= c(m_2+1,n) - c(m_2+1,n-1) \text{ by Proposition 3.1} \\
&= \sum_{k=0}^{1} (-1)^{1-k} c(m_1,n-1+k) \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{1}{m-k} c(m_1,n-m+k) \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{m-k} c(m_1,n-m+k)
\end{aligned} \tag{10}$$

The next two calculations verifies (6) for the base case.

If $n = m + 1 = 2$ then

$$\begin{aligned}
c(m_2,n) &= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{m-k}c(m_1,n-m+k) \\
&= \sum_{k=0}^{1}(-1)^{1-k}\binom{1}{1-k}c(m_1,2-1+k) \\
&= \sum_{k=0}^{1}(-1)^{1-k}\binom{1}{1-k}c(m_1,k+1) \\
&= \sum_{k=1}^{2}(-1)^{2-k}\binom{1}{2-k}c(m_1,k) \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{m}{n-k}c(m_1,k)
\end{aligned} \tag{11}$$

If $n = m = 1$ then

$$\begin{aligned}
c(m_2,n) &= c(m_1,1) \\
&= (-1)^0\binom{1}{0}c(m_1,1) \\
&= \sum_{k=1}^{1}(-1)^{1-k}\binom{1}{1-k}c(m_1,k) \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{m}{n-k}c(m_1,k)
\end{aligned} \tag{12}$$

Next, the induction step on m is performed. Assume that $m = m_1 - m_2 > 1$, and that the equations hold for $m - 1$. Also, assume as a base case for a second induction argument that $n = 1$.

$$\begin{aligned}
c(m_1,n) &= c(m_2,1) \text{ by Proposition 3.1} \\
&= \binom{m-1}{m-1}c(m_2,1) \\
&= \sum_{k=1}^{n}\binom{m+1-1-1}{m-1}c(m_2,k) \\
&= \sum_{k=1}^{n}\binom{m+n-k-1}{m-1}c(m_2,k)
\end{aligned} \tag{13}$$

Notice that when $n = 1$, $n \geq m + 1$ is never true so only (6) needs to be verified.

$$\begin{aligned}
c(m_2,n) &= c(m_1,1) \\
&= (-1)^0\binom{m}{0}c(m_1,1) \\
&= \sum_{k=1}^{1}(-1)^{1-1}\binom{m}{1-1}c(m_1,k) \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{m}{n-k}c(m_1,k)
\end{aligned} \tag{14}$$

Finally the induction step for n is performed. Assume (4), (5), and (6) hold for $n-1$ when $n>1$.

$$\begin{aligned} c(m_1,n) =& c(m_1,n-1)+c(m_1-1,n) \\ =& \sum_{k=1}^{n-1}\binom{m+(n-1)-k-1}{m-1}c(m_2,k) \\ &+\sum_{k=1}^{n}\binom{(m-1)+n-k-1}{(m-1)-1}c(m_2,k) \\ =& \sum_{k=1}^{n-1}\binom{m+n-k-2}{m-1}c(m_2,k) \\ &+\sum_{k=1}^{n-1}\binom{m+n-k-2}{m-2}c(m_2,k)+c(m_2,n) \\ =& \sum_{k=1}^{n-1}[\binom{m+n-k-2}{m-1}+\binom{m+n-k-2}{m-2}]c(m_2,k)+c(m_2,n) \\ =& \sum_{k=1}^{n-1}\binom{m+n-k-2+1}{m-1}c(m_2,k)+c(m_2,n) \text{ by Lemma 3.2} \\ =& \sum_{k=1}^{n}\binom{m+n-k-1}{m-1}c(m_2,k) \end{aligned} \tag{15}$$

When $n \geq m+1$,

$$\begin{aligned} c(m_2,n) =& c(m_2+1,n)-c(m_2+1,n-1) \\ =& \sum_{k=0}^{m-1}(-1)^{(m-1)-k}\binom{m-1}{(m-1)-k}c(m_1,n-(m-1)+k) \\ &-\sum_{k=0}^{m-1}(-1)^{(m-1)-k}\binom{m-1}{(m-1)-k}c(m_1,(n-1)-(m-1)+k) \\ =& \sum_{k=0}^{m-1}(-1)^{m-k-1}\binom{m-1}{m-k-1}c(m_1,n-m+k+1) \\ &+\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m-1}{m-k-1}c(m_1,n-m+k) \\ =& \sum_{k=1}^{m}(-1)^{m-k}\binom{m-1}{m-k}c(m_1,n-m+k) \\ &+\sum_{k=0}^{m-1}(-1)^{m-k}\binom{m-1}{m-k-1}c(m_1,n-m+k) \\ =& \sum_{k=1}^{m-1}(-1)^{m-k}[\binom{m-1}{m-k}+\binom{m-1}{m-k-1}]c(m_1,n-m+k) \\ &+c(m_1,n)+(-1)^{m}c(m_1,n-m) \\ =& \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{m-k}c(m_1,n-m+k) \end{aligned} \tag{16}$$

Verification of (6) is split into two cases.

When $n = m + 1$,

$$\begin{aligned} c(m_2, n) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{m-k} c(m_1, n-m+k) \\ &= \sum_{k=0}^{n-1} (-1)^{n-k-1} \binom{m}{n-k-1} c(m_1, k+1) \\ &= \sum_{k=1}^{n} (-1)^{n-k} \binom{m}{n-k} c(m_1, k) \end{aligned} \tag{17}$$

When $n < m + 1$,

$$\begin{aligned} c(m_2, n) =& c(m_2+1, n) - c(m_2+1, n-1) \\ =& \sum_{k=1}^{n} (-1)^{n-k} \binom{m-1}{n-k} c(m_1, k) \\ &- \sum_{k=1}^{n-1} (-1)^{(n-1)-k} \binom{m-1}{(n-1)-k} c(m_1, k) \\ =& \sum_{k=1}^{n} (-1)^{n-k} \binom{m-1}{n-k} c(m_1, k) + \sum_{k=1}^{n-1} (-1)^{n-k} \binom{m-1}{n-k-1} c(m_1, k) \\ =& c(m_1, n) + \sum_{k=1}^{n-1} (-1)^{n-k} [\binom{m-1}{n-k} c(m_1, k) + \binom{m-1}{n-k-1}] c(m_1, k) \\ =& \sum_{k=1}^{n} (-1)^{n-k} \binom{m}{n-k} c(m_1, k) \end{aligned} \tag{18}$$

□

3.3 Notable Corollaries

In general, there is no way to determine the exact distance between two sequences that occur in a summation chain. Under certain conditions, however, Corollary 3.4 determines the location of sequences in relation to each other within a summation chain. This corollary is essential in determining whether two arbitrary sequences appear in the same summation chain.

Corollary 3.4. *Let M be a $\mathbb{Z}$-Module such that the order of every nonzero element in M is infinite. Let c be a non-trivial Σ-chain in M. Let $N = min\{n \in \mathbb{N} : c(0,n) \neq 0\}$. Let $m^*, m_1, m_2 \in \mathbb{Z}$.*

$$m^* \cdot c(m_1, N) = c(m_2, N+1) - c(m_1, N+1) \tag{19}$$

if and only if $m^* = m_2 - m_1$.

Remark 3.3. When M is a field with unity 1_M, the equation (19) can be written,

$$m^* \cdot (1_M) = \frac{c(m_2, N+1) - c(m_1, N+1)}{c(m_1, N)} \tag{20}$$

Proof. Suppose $m^* = m_2 - m_1$. If $m_1 = m_2$ both sides of the equation are zero.

If $m_1 < m_2$ then by Theorem 3.3,

$$\begin{aligned} c(m_2, N+1) &= \sum_{k=1}^{N+1} \binom{m_2 - m_1 + N + 1 - k - 1}{m_1 - m_2 - 1} c(m_1, k) \\ &= \binom{m_2 - m_1}{m_2 - m_1 - 1} c(m_1, N) + \binom{m_2 - m_1 - 1}{m_2 - m_1 - 1} c(m_1, N+1) \\ &= (m_2 - m_1) c(m_1, N) + c(m_1, N+1) \end{aligned} \tag{21}$$

If $m_1 > m_2$ and $n \geq m_1 - m_2 + 1$, then by Theorem 3.3,

$$\begin{aligned} c(m_2, N+1) &= \sum_{k=0}^{m_1 - m_2} (-1)^{m_1 - m_2 - k} \binom{m_1 - m_2}{m_1 - m_2 - k} c(m_1, N - (m_1 - m_2) + k - 1) \\ &= -\binom{m_1 - m_2}{1} c(m_1, N) + \binom{m_1 - m_2}{0} c(m_1, N+1) \\ &= (m_2 - m_1) c(m_1, N) + c(m_1, N+1) \end{aligned} \tag{22}$$

If $m_1 > m_2$ and $n \leq m_1 - m_2 + 1$, then by Theorem 3.3,

$$\begin{aligned} c(m_2, N+1) &= \sum_{k=1}^{N+1} (-1)^{n-k} \binom{m_1 - m_2}{N - k + 1} c(m_1, k) \\ &= -\binom{m_1 - m_2}{1} c(m_1, N) + \binom{m_1 - m_2}{0} c(m_1, N+1) \\ &= (m_2 - m_1) c(m_1, N) + c(m_1, N+1) \end{aligned} \tag{23}$$

Therefore,

$$(m_2 - m_1) c(m_1, N) = c(m_2, N+1) - c(m_1, N+1) \tag{24}$$

Suppose (19) holds for an integer m^*. Then by (24),

$$m^* \cdot c(m_1, N) = c(m_2, N+1) - c(m_1, N+1) = (m_2 - m_1) c(m_1, N) \tag{25}$$

This implies that $(m^* - m_2 + m_1) c(m_1, N) = 0$. Since $c(m_1, N) \neq 0$, $c(m_1, N)$ has infinite order so $m^* - m_2 + m_1 = 0$. Therefore, $m^* = m_2 - m_1$. □

In the case of complex valued summation chains, each chain never repeats a sequence unless the chain is trivial. This property comes from the fact that zero is the only complex number that has finite order as stated by Corollary 3.5.

Corollary 3.5. *Let M be a $\mathbb{Z}$-Module such that the order of every nonzero element in M is infinite. Let c be a Σ-chain in M. Let $m_1, m_2 \in \mathbb{Z}$ such that $m_1 \neq m_2$. If $c(m_1, n) = c(m_2, n), \forall n \in \mathbb{N}$, then c is the trivial chain.*

Proof. Assume c is not the trivial chain. Let $N = min\{n \in \mathbb{N} : c(0, n) \neq 0\}$. By Corollary 3.4, $(m_2 - m_1)c(m_1, N) = 0$. Since $c(m_1, N)$ is nonzero, its order is infinite. Therefore, for this equation to hold $m_2 = m_1$ which is a contradiction. □

It has not been determined whether or not Corollary 3.5 is true for all summation chains of sequences.

4. Conclusion

Editor's note: This paper represents the introduction of the concept of summation chains of complex valued sequences. Key concepts and definitions that set the stage for further results in the development of this theory. Part 2 will discuss sum-related chains, and handling leading zeros.

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