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Masters Thesis

Summation Chains of Sequences Part 2: Relationships between Sequences via Summation Chains

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Abstract
(Editor's note:) This paper represents the second installment of a masters thesis by Jonathan Johnson. This paper continues the development of the theory of summation chains of sequences. The concept of sum-related is defined: two sequences are sum-related if one sequence appears in the summation chain of the other. The main result is a theorem to determine if two given sequences are sum-related.

Keywords
sequences, summation chains

Contents

Introduction

(Editor's note:)Previously, in [5], the concept of a summation chain of sequences was defined. We reproduce the example and introduction here: Given a complex-valued sequence $(a_n)_{n=1}^{\infty}$, the *sequence of partial sums* of (a_n) is given by the sequence $(a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots, \sum_{i=1}^{n} a_i, \ldots)$. The *sequence of differences* of (a_n) is given by the sequence $(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_n - a_{n-1}, \ldots)$. The processes of finding the sequence of partial sums and finding the sequence of differences of a sequence are inverses of each other so every sequence is the sequence of differences of its sequence of partial sums and the sequence of partial sums of its sequence of differences. Every sequence has a unique sequence of partial sums and a unique sequence of differences so it is always possible to find the sequence of partial sums of the sequence of partial sums and repeat the process ad infinitum. Similarly, we can find the sequence of differences of the sequence of differences and repeat ad infinitum. The result is a doubly infinite sequence or "chain" of sequences where each sequence is the sequence of partial sums of the previous sequence and the sequence of differences of the following sequence.

Example 0.1. Let $a^{(0)}$ be the sequence defined by $a_n^{(0)} = (-1)^{n+1}$, for all $n \in \mathbb{N}$. For all integers $m > 0$, let $a^{(m)}$ be the sequence of partial sums of $a^{(m-1)}$, and for all integers $m < 0$, let $a^{(m)}$ be the sequence of differences of $a^{(m+1)}$.

Every sequence can be used to create a unique chain of sequences.

Editor's note: This installment examines the relationship between sequences and their summation chains, and provides a way to determine whether two sequences are sum-related.

1. Summation Chains and Sequences

1.1 Sum-Related Chains

Consider the following sequences with their Σ-chains.

$s_1 = (1,0,0,0,\ldots)$
$s_2 = (1,1,1,1,\ldots)$
$s_3 = (0,1,0,0,\ldots)$

These chains, while distinct, contain the same pattern of numbers. The entries of $\Phi_{\Sigma[\mathbb{R}]}(s_2)$ are the entries of $\Phi_{\Sigma[\mathbb{R}]}(s_1)$ moved up. This occurs because s_2 is the sequence of partial sums of s_1. The entries of $\Phi_{\Sigma[\mathbb{R}]}(s_3)$ are the entries of $\Phi_{\Sigma[\mathbb{R}]}(s_1)$ with an extra column of zeros. The remainder of this section focuses on these relationships.

Definition 1.1. Let M be a $\mathbb{Z}$-Module. Let (a_n) and (b_n) be sequences in M. (a_n) is *sum-related* to (b_n), denoted $(a_n) \overset{\Sigma}{\sim} (b_n)$, if (a_n) is a sequence in $\Phi_\Sigma((b_n))$.

If two sequences are sum-related, then one sequence can be obtained by finding the sequences of partial sums of the sequences of partial sums of the other sequence.

Remark 1.1. The trivial sequence $(0,0,0,\ldots)$ is sum-related only to itself.

Proposition 1.1. *Sum relation defines an equivalence class.*

Proof. This proposition follows from Proposition 2.1 of[5]. □

Proposition 1.2. *Let c and d be Σ-chains. If there exists $m \in \mathbb{Z}$ such that $c(m,n) = d(m,n)$, for all $n \in \mathbb{N}$, then $c = d$.*

Proof. This proposition follows from Proposition 2.2 of [5]. □

Proposition 1.3. *Let M be a $\mathbb{Z}$-Module. Let c be a Σ-chain in M. Let $d : \mathbb{Z} \times \mathbb{N} \to M$ be defined such that there exists $k \in \mathbb{Z}$ such that $d(m,n) = c(m+k,n)$ for every $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, then d is a Σ-chain, and every sequence in c is sum-related to every sequence in d.*

Proof. This proposition follows from Proposition 2.3 of [5] □

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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
$a^{(2)}$	1	1	2	2	3	3	4	4	⋯
$a^{(1)}$	1	0	1	0	1	0	1	0	⋯
$a^{(0)}$	1	−1	1	−1	1	−1	1	−1	⋯
$a^{(-1)}$	1	−2	2	−2	2	−2	2	−2	⋯
$a^{(-2)}$	1	−3	4	−4	4	−4	4	−4	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure 0.1. Example 0.1

m \ n	1	2	3	4	⋯
⋮	⋮	⋮	⋮	⋮	
2	1	2	3	4	⋯
1	1	1	1	1	⋯
0	1	0	0	0	⋯
-1	1	-1	0	0	⋯
-2	1	-2	1	0	⋯
⋮	⋮	⋮	⋮	⋮	

$\Phi_{\Sigma[\mathbb{R}]}(s_1)$

m \ n	1	2	3	4	⋯
⋮	⋮	⋮	⋮	⋮	
2	1	3	6	10	⋯
1	1	2	3	4	⋯
0	1	1	1	1	⋯
-1	1	0	0	0	⋯
-2	1	-1	0	0	⋯
⋮	⋮	⋮	⋮	⋮	

$\Phi_{\Sigma[\mathbb{R}]}(s_2)$

m \ n	1	2	3	4	⋯
⋮	⋮	⋮	⋮	⋮	
2	0	1	2	3	⋯
1	0	1	1	1	⋯
0	0	1	0	0	⋯
-1	0	1	-1	0	⋯
-2	0	1	-2	1	⋯
⋮	⋮	⋮	⋮	⋮	

$\Phi_{\Sigma[\mathbb{R}]}(s_3)$

Figure 1.1. Chains of s_1, s_2, s_3

Definition 1.2. Let c be a Σ-chain, and let $k \in \mathbb{Z}$. The Σ-chain c^k is defined for all integers m and natural numbers n, by $c^k(m,n) = c(m+k,n)$. c^k is said to be the Σ-chain c *shifted by* k.

Remark 1.2. If s_1 and s_2 are sequences generating Σ-chains c_1 and c_2 respectively, then saying $s_1 \overset{\Sigma}{\sim} s_2$ is equivalent to saying $c_1 = c_2^k$ for some integer k.

When do two sequences appear in the same summation chain? By Definition 1.1, this is the same as asking when sequences are sum-related. Consider the sequence $(x_k) = (0,0,1,-1,1,-1,\ldots)$. The sequences sum-related to (x_k) are the sequences that are in $\Phi_{\Sigma[\mathbb{R}]}((x_k))$. (Figure 1.2) Notice that each sequence in the chain starts with two zeros followed by a one. The sequences $(0,1,10,20,30,\ldots)$ and $(0,0,4,5,6,\ldots)$ could not be sum-related to (x_k) since they do not begin with this sequence. This strategy can eliminate most pairs of sequences.

Lemma 1.4. *Given two nontrivial sequences (a_k) and (b_k) such that $(a_k) \overset{\Sigma}{\sim} (b_k)$, define N_a and N_b to be the indices of the first nonzero terms of (a_k) and (b_k) respectively, then N_a=N_b, and $a_{N_a} = b_{N_b}$.*

m \ n	1	2	3	4	5	6	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
2	0	0	1	1	2	2	⋯
1	0	0	1	0	1	0	⋯
0	0	0	1	-1	1	-1	⋯
-1	0	0	1	-2	2	-2	⋯
-2	0	0	1	-3	4	-4	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	

Figure 1.2. $\Phi_{\Sigma[\mathbb{R}]}((x_k))$

Proof. Let $c_b = \Phi_\Sigma((b_k))$. For a specific integer m, $a_n = c_b(m,n)$ for every $n \in \mathbb{N}$. If $m = 0$, then the lemma's conclusion clearly follows. When $m \neq 0$, since similarity and equality are symmetric, it can be assumed that m is positive.

By Lemma 1.6, for all $k < N_b$, $a_k = c_b(m,k) = 0$.

$$\begin{aligned} a_{N_b} &= \sum_{k=1}^{N_b} \binom{m+N_b-k-1}{m-1} c_b(0,k) \text{ by Theorem 3.3 of [5]} \\ &= \sum_{k=1}^{N_b} \binom{m+N_b-k-1}{m-1} b_k \\ &= b_{N_b} \end{aligned} \tag{1}$$

Since $b_{N_b} \neq 0$, $a_{N_b} \neq 0$; therefore, N_a=N_b, and $a_{N_a} = b_{N_b}$. ☐

In general, after exploiting Lemma 1.4, it is difficult to determine sum relation of sequences. It is not possible to exhaustively check every sequence in a chain. However, when every nonzero element in the $\mathbb{Z}$-Module has infinite order, Corollary 3.4 of [5] tells where to look for sum-related sequences. Thus in this case, a closed form procedure for determining sum relation can be used.

Theorem 1.5. *Let M be a $\mathbb{Z}$-Module such that the order of every nonzero element in M is infinite. Let (a_n) and (b_n) be sequences in M, and let $c_a = \Phi_\Sigma((a_n))$. Let N_1 be the index of the first nonzero term of (a_n), and let N_2 be the index of the first nonzero term of (b_n). $(a_n) \overset{\Sigma}{\sim} (b_n)$ if and only if all of the following conditions hold:*

(i) $N_1 = N_2$, if this is true let $N = N_1 = N_2$

(ii) $a_N = b_N$,

(iii) There exists $m \in \mathbb{Z}$ such that $m \cdot a_N = b_{N+1} - a_{N+1}$,

(iv) $b_n = c_a(m,n), \forall n \in \mathbb{N}$.

Remark 1.3. When the integer m exists in Theorem 1.5, it is unique by Corollary 3.4 of [5].

Proof. Assume $(a_n) \overset{\Sigma}{\sim} (b_n)$. (i) and (ii) follow from Lemma 1.4. There exists integer m such that (iv) holds by definition of sum relation. By Corollary 3.4 of [5], (iii) holds as follows:

$$\begin{aligned} m \cdot a_N &= m \cdot c_a(0,N) \\ &= c_a(m,N+1) - c_a(0,N+1) \\ &= b_{N+1} - a_{N+1} \end{aligned} \tag{2}$$

Assume that conditions (i),(ii),(iii), and (iv) hold. The existence of integer m in (iii) along with (iv) implies that $(a_n) \overset{\Sigma}{\sim} (b_n)$. ☐

Example 1.1. Let $M = \mathbb{Z}$.
Let $(a_n) = (1,-1,-1,1,0,0,0,0,0,\ldots)$ and $b_n = n^2$, $\forall n \in \mathbb{N}$.
Let $c_a = \Phi_\Sigma((a_n))$.

The index of the first nonzero terms of both sequences is 1. Also, $a_1 = b_1 = 1$ so (i) and (ii) are satisfied. $b_2 - a_2 = 5 = 5 \cdot a_1$ so (iii) holds. If there is a chain relation, (b_n) will be the fifth sequence in c_a. (iv) has already been shown for $n = 1$ and $n = 2$. Using formulas from Theorem 3.3 of [5], the computation for the other indices is as follows:

$$c_a(5,3) = \binom{6}{4}(1) + \binom{5}{4}(-1) + \binom{4}{4}(-1) = 15 - 5 - 1 = 9 = b_3 \tag{3}$$

For $n \geq 4$,

$$\begin{aligned} c_a(5,n) &= \sum_{k=1}^{n} \binom{4+n-k}{4} a_k \\ &= \binom{n+3}{4}(1) + \binom{n+2}{4}(-1) + \binom{n+1}{4}(-1) + \binom{n}{4}(1) \\ &= \frac{(n+3)(n+2)(n+1)n}{24} - \frac{(n+2)(n+1)n(n-1)}{24} \\ &\quad - \frac{(n+1)n(n-1)(n-2)}{24} + \frac{n(n-1)(n-2)(n-3)}{24} \\ &= n^2 \\ &= b_n \end{aligned} \tag{4}$$

Therefore, by Theorem 1.5, $(a_n) \overset{\Sigma}{\sim} (b_n)$.

Example 1.2. Let $M = \mathbb{R}$.
Let $a_n = \frac{1}{n}$ and $b_n = n$, $\forall n \in \mathbb{N}$.
Let $c_a = \Phi_\Sigma((a_n))$.

The index of the first nonzero terms of both sequences is 1. Also, $a_1 = b_1 = 1$ so (i) and (ii) are satisfied. However $b_2 - a_2 = 1.5$ which cannot be obtained by multiplying an integer by a_1. Therefore, by Theorem 1.5, $(a_n) \overset{\Sigma}{\nsim} (b_n)$.

Example 1.3. Let $M = \mathbb{Z}$. Let $a_n = n^2 - 3n + 2$ and $b_n = -\frac{n^4+3n^3+14n^2-48n+32}{7}$, $\forall n \in \mathbb{N}$.
Let $c_a = \Phi_\Sigma((a_n))$.

The index of the first nonzero terms of both sequences is 3. Also, $a_3 = b_3 = 2$ so (i) and (ii) are satisfied. $b_4 - a_4 = -6 = -3 \cdot a_3$ so (iii) holds. If there is a chain relation, (b_n) will be the third sequence down in c_a. However, $c_a(-3,5) = 0$ while $b_5 = -\frac{108}{7}$. Therefore, by Theorem 1.5, $(a_n) \overset{\Sigma}{\nsim} (b_n)$.

Example 1.4. Let $M = \mathcal{Z}[X]$.
Let $a_n = \sum_{i=0}^{n-1}(i+1)x^{n-i}$, $\forall n \in \mathbb{N}$, and let $b_1 = x$ and $b_n = x^n - x^{n-1}$, $n \geq 2$.
Let $c_a = \Phi_\Sigma((a_n))$.

The index of the first nonzero terms of both sequences is 1. Also, $a_1 = b_1 = x$ so (i) and (ii) are satisfied. $b_2 - a_2 = (x^2 - x) - (x^2 + 2x) = -3x = -3 \cdot a_1$ so (iii) holds. If there is a chain relation, (b_n) will be the third sequence down in c_a. (iv) has already been shown for $n = 1$ and $n = 2$.

$$\begin{aligned} c_a(-3,3) &= \binom{3}{1}(x) - \binom{3}{2}(x^2+2x) + \binom{3}{3}(x^3+2x^2+3x) \\ &= 3x + 3(x^2+2x) + x^3 + 2x^2 + 3x \\ &= x^3 - x^2 \\ &= b_3 \end{aligned} \tag{5}$$

For $n \geq 4$,

$$\begin{aligned}
c_a(-3,n) &= \sum_{k=0}^{3}(-1)^{3-k}\binom{3}{3-k}a_{n-3+k} \\
&= -\binom{3}{0}\sum_{i=0}^{n-4}(i+1)x^{n-i-3} + \binom{3}{1}\sum_{i=0}^{n-3}(i+1)x^{n-i-1} \\
&\quad -\binom{3}{2}\sum_{i=0}^{n-2}(i+1)x^{n-i-1} + \binom{3}{3}\sum_{i=0}^{n-1}(i+1)x^{n-i} \\
&= -\sum_{i=3}^{n-1}(i-2)x^{n-i} + 3\sum_{i=2}^{n-1}(i-1)x^{n-i} - 3\sum_{i=1}^{n-1}ix^{n-i} + \sum_{i=0}^{n-1}(i+1)x^{n-i} \\
&= -\sum_{i=3}^{n-1}(i-2)x^{n-i} + 3\sum_{i=3}^{n-1}(i-1)x^{n-i} - 3\sum_{i=3}^{n-1}ix^{n-i} + \sum_{i=3}^{n-1}(i+1)x^{n-i} \\
&\quad + 3x^{n-2} - 3x^{n-1} - 6x^{n-2} + x^n + 2x^{n-1} + 3x^{n-2} \\
&= \sum_{i=3}^{n-1}[-(i-2) + 3(i-1) - 3i(i+1)]x^{n-i} + x^n - x^{n-1} \\
&= x^n - x^{n-1} \\
&= b_n
\end{aligned} \tag{6}$$

Therefore, by Theorem 1.5, $(a_n) \overset{\Sigma}{\sim} (b_n)$.

1.2 Chains of Sequences with Leading Zeros

Adding zeros to the beginning of a sequence adds columns of zeros to be beginning of the sequences summation chain.

Lemma 1.6. *Let c be a nontrivial Σ-chain in M. Let $N = min\{n \in \mathbb{N} : c(0,n) \neq 0\}$. For every integer, m, and for every natural number, $n < N$, $c(m,n) = 0$.*

Proof. Let $m \in \mathbb{Z}$ and let $n \in \mathbb{N}$ such that $n < N$.

Clearly $c(0,n) = 0$.
If $m > 0$ then

$$c(m,n) = \sum_{k=1}^{n}\binom{m+n-k-1}{m-1}c(0,k) = 0 \tag{7}$$

If $m < 0$ then let $m' = -m$.
If $n \geq m' + 1$,

$$c(m,n) = \sum_{k=0}^{m'}(-1)^{m'-k}\binom{m'}{m'-k}c(0,n-m'+k) = 0 \tag{8}$$

If $n \leq m' + 1$,

$$c(m,n) = \sum_{k=1}^{n}(-1)^{n-k}\binom{m'}{n-k}c(0,k) = 0 \tag{9}$$

□

Proposition 1.7. *Let M be a $\mathbb{Z}$-Module. Let c be a nontrivial Σ-chain in M. Let the natural number $N = min\{n \in \mathbb{N} : c(0,n) \neq 0\}$. Let $d : \mathbb{Z} \times \mathbb{N} \to M$ be defined such that $d(m,n) = c(m,n+N-1)$ for every $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then d is a Σ-chain.*

Proof. Let $m \in \mathbb{Z}$, and let $n \in \mathbb{N}$. By Lemma 1.6, $c(m,k) = 0$ for $k < N$.

$$\begin{aligned}
d(m,n) &= c(m,n+N-1) \\
&= \sum_{k=1}^{n+N-1} c(m-1,k) \\
&= \sum_{k=N}^{n+N-1} c(m-1,k) \\
&= \sum_{k=1}^{n} c(m-1,k-N+1) \\
&= \sum_{k=1}^{n} d(m-1,k)
\end{aligned} \tag{10}$$

□

Corollary 1.8. *Let M be a $\mathbb{Z}$-Module, and let (a_n) and (b_n) be sequences in M, such that for a positive integer k, $b_{n+k} = a_n$ for all natural numbers n and $b_n = 0$ when $n \leq k$. Let $c_a = \Phi_\Sigma((a_n))$ and $c_b = \Phi_\Sigma((a_b))$. For every integer m and natural number n, $c_b(m,n+k) = c_a(m,n)$, and $c_b(m,n) = 0$ when $n < k$.*

Proof. Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. When $m = 0$ then $c_b(0,n+k) = b_{n+k} = a_n = c_a(0,n)$.

Assume that the result holds for m,

$$\begin{aligned}
c_b(m+1,n+k) &= \sum_{i=1}^{n+k} c_b(m,i) \\
&= \sum_{i=k+1}^{n+k} c_b(m,i) \\
&= \sum_{i=1}^{n} c_b(m,i+k) \\
&= \sum_{i=1}^{n} c_a(m,i) \\
&= c_a(m+1,n)
\end{aligned} \tag{11}$$

$$\begin{aligned}
c_b(m-1,n+k) &= c_b(m,n+k) - c_b(m,n+k-1) \\
&= c_a(m,n) - c_a(m,n-1) \\
&= c_a(m-1,n)
\end{aligned} \tag{12}$$

Therefore, the result follows by induction. □

2. Conclusion

Editor's note: This paper represents the second installment of the concept of summation chains of complex valued sequences. Here, a new way to determine the relationship between two sequences was defined. The concept of sum-relation was defined, and a theorem provided a way to determine the relationship between two sequences. Part 3 will discuss sequence chains generated by linear function rules

References

[1] Apostol, T., 1974, *Mathematical Analysis 2nd Edition*, Addison-Wesley Publishing Company

[2] Brin, M. and Stuck, G., 2002, *Introduction to Dynamical Systems*, Cambridge University Press

[3] Cameron, P. J., 1992, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press

[4] Cook, R. G., 2014, *Infinite Matrices and Sequence Spaces*, Dover Publications, Inc.

[5] Johnson, J. (2017)*Summation Chains of Sequences Part 1: Introduction, Generation, and Key Definitions*AACTO, Vol. 2, Issue 2. https://www.theresearchcortex.com/summationchainsofseqpart1

[6] Lee, J., 2011, *Introduction to Topological Manifolds*, Springer Science+Business Media LLC

[7] Leon, S. J., 2002, *Linear Algebra with Applications 6th Edition*, Prentice Hall, Inc.

[8] Scheinerman, E. R., 1996, *Invitation to Dynamical Systems*, Dover Publications, Inc.