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Masters Thesis

Summation Chains of Sequences
Part 3: Sequence Chains from Linear Functions

Jonathan C. Johnson*

Abstract
(Editor's note:) This paper represents the third installment of a masters thesis by Jonathan Johnson. This paper continues the development of the theory of summation chains of sequences. Since summation chains are doubly infinite, it's important to know how little information we actually need to define a chain. The linearity of the function rules that generates a summation chain helps to answer this question. The notion of *uniquely completable* is defined from the set of positions, and several important theorems are developed to determine when a set of positions is uniquely completeable.

Keywords
sequences, summation chains

Contents

Introduction

(Editor's note:) In Section 2 of [5], Johnson notes that summation chains can be generated by the function rule $T_\Sigma : M^\infty \to M^\infty$,

$$T_\Sigma(x_1, x_2, x_3, \ldots) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots)$$

where M^∞ is a $\mathbb{Z}$-Module. This is a formal way to define the operator that generates the partial sums of a given sequence. Johnson proves in this installment that this function rule is a linear operator. The linearity of these function operators assists in determining how little information we need to know about the chain to be able to uniquely define it.

Johnson introduces the notion of the set of positions, which becomes the smallest amount of information that can define a summation chain under certain conditions.

1. Chains Generated by Linear Function Rules

Since vector-valued chains are vector-valued functions, they inherit the scalar multiplication and addition operations defined for functions on vector spaces. The linearity of the summation chain function

*University of Texas at Austin

rule leads to many interesting and useful results. The first of these is the closure of the set of summation chains under linear operations. This result holds for any function rule that is a linear operator.

Lemma 1.1. *Let M be a an 1-dimensional vector space with scalar field F. The summation function rule T_Σ defined in Remark 3.1 of [6] is a linear operator on M^∞.*

Proof. Let (x_n) and (y_n) be M valued sequences, and let a and b be in M.

$$\begin{aligned} T_\Sigma(a\cdot(x_n)+b\cdot(y_n)) &= T_\Sigma(ax_1+by_1, ax_2+by_2, ax_3+by_3,\dots) \\ &= (ax_1+by_1, ax_1+ax_2+by_1+by_2,\dots) \\ &= a\cdot(x_1, x_1+x_2,\dots)+b\cdot(y_1, y_1+y_2,\dots) \\ &= aT_\Sigma(x_n)+bT_\Sigma(y_n) \end{aligned} \tag{1}$$

□

Proposition 1.2. *Let $T: V\to V$ be a function rule where V is a vector space with scalar field F. $\mathcal{C}_T$ is closed under scalar multiplication and addition if and only if T is a linear operator.*

Proof. Let $a, b\in F$, let $u, v\in V$, and let $c_1, c_2\in\mathcal{C}_T$.

Assume $\mathcal{C}_T$ is closed under scalar multiplication and addition. Let $c_u=\Phi_T(u)$ and $c_v=\Phi_T(v)$. Also let $c_{uv}=ac_u+bc_v$ so $c_{uv}\in\mathcal{C}_T$.

$$\begin{aligned} T(au+bv) &= T(ac_u(0)+bc_v(0)) \\ &= T(c_{uv}(0)) \\ &= c_{uv}(1) \\ &= ac_u(1)+bc_v(1) \\ &= aT(u)+bT(v) \end{aligned} \tag{2}$$

So T is a linear operator.

Assume T is a linear operator. Let $c_3=ac_1+bc_2$. Let $m\in\mathbb{Z}$.

$$\begin{aligned} c_3(m) &= ac_1(m)+bc_2(m) \\ &= aT(c_1(m-1))+bT(c_2(m-1)) \\ &= T(ac_1(m-1)+bc_2(m-1)) \\ &= T(c_3(m-1)) \end{aligned} \tag{3}$$

So $c_3\in\mathcal{C}_T$. □

Corollary 1.3. *Let $T: V\to V$ be a bijective linear operator on vector space V with scalar field F, then $\mathcal{C}_T$ is a subspace of the space of all functions from $\mathbb{Z}$ to V,$V^{\mathbb{Z}}$, Φ_T is an isomorphism, and $\mathcal{C}_T\cong V$.*

Proof. Clearly $\mathcal{C}_T\subset V^{\mathbb{Z}}$, and $\mathcal{C}_T$ is closed under vector operations by Proposition 1.2; therefore, $\mathcal{C}_T$ is a subspace of $V^{\mathbb{Z}}$.

Let $a, b\in F$, let $u, v\in V$, and let $m\in\mathbb{Z}$.

$$\begin{aligned} (\Phi_T(au+bv))(m) &= T^m(au+bv) \\ &= aT^m(u)+bT^m(v) \\ &= a(\Phi_T(u))(m)+b(\Phi_T(v))(m) \end{aligned} \tag{4}$$

This means that $\Phi_T(au+bv)=a\Phi_T(u)+b\Phi_T(v)$ so Φ_T is a homomorphism. Since Φ_T is bijective by Remark 2.1 of [6], Φ_T is an isomorphism from V to $\mathcal{C}_T$. □

Corollary 1.4. *Let M be an 1-dimensional vector space with scalar field F. $\mathcal{C}_{\Sigma[M]}$ is a vector space, and $\Phi_{\Sigma[M]}$ is an isomorphism from M^∞ to $\mathcal{C}_{\Sigma[M]}$.*

Proof. Follows from Lemma 1.1 and Corollary 1.3. □

2. Sets of Positions and Completeability

Viewing the summation operator as a linear operator is useful for answering the question on how little information must be known to define a chain.

Definition 2.1. A *set of positions*, U, is a nonempty subset of $\mathbb{Z} \times \mathbb{N}$.
Let $N \in \mathbb{N}$, $U_N = \{(m,n) \in U : n \leq N\}$.

Definition 2.2. Let M be an 1-dimensional vector space with scalar field F. Let $T : M^\infty \to M^\infty$ be a bijective linear operator. A set of positions, U, is *(uniquely) completable for* T if for every function $d : U \to M$ there exists a (unique) T-chain, c, such that $c(m,n) = d(m,n)$ for all $(m,n) \in U$.

To say that a set of positions U is uniquely compleatable for T is the same as saying that every T-chain can be defined by defining each of the positions in U.

Lemma 2.1. *Let M be an 1-dimensional vector space with scalar field F. Let $T : M^\infty \to M^\infty$ be a bijective linear operator. Let U be a set of positions, U is completable for T if and only if every nonempty subset of U is completable for T.*

Proof. Assume U is completable. Let V be a nonempty subset of U, and let $d : V \to M$.
Define $\tilde{d} : U \to M$ such that $\tilde{d}(m,n) = d(m,n)$ when $(m,n) \in V$ and zero otherwise. Since U is completable, there exists T-chain, c, such that $c(m,n) = \tilde{d}(m,n)$ for all $(m,n) \in U$. Since $V \subseteq U$, $c(m,n) = \tilde{d}(m,n) = d(m,n)$ for all $(m,n) \in V$ so V is completable for T.
Assume every nonempty subset of U is completable, then U is completable since $U \subseteq U$. □

Lemma 2.2. *Let M be an 1-dimensional vector space with scalar field F. Let $T : M^\infty \to M^\infty$ be a bijective linear operator. Let U be a set of positions, and let V be a set of positions containing U. If U is uniquely completable for T and V is completable for T, then V is uniquely completable for T.*

Proof. Let $d : V \to M$. Let c_1 and c_2 be T-chains such that $c_1(m,n) = c_2(m,n) = d(m,n)$ for all $(m,n) \in V$. Consider $d|_U : U \to M$. $c_1(m,n) = c_2(m,n) = d|_U(m,n)$ for all $(m,n) \in U$. Since U is uniquely completable, c_1=c_2. Therefore, V is uniquely completable. □

How much information is needed to define a summation chain? The summation function rule has the additional property that it can be represented by an infinite-dimensional lower-triangular matrix. The question of how much information is need to define a summation chain will be addressed by studying matrices of this form. First, an ordering of the elements of $\mathbb{Z} \times \mathbb{N}$ is defined in order to create a consistent indexing of sets of positions.

Definition 2.3. Let (m_1,n_1) and (m_2,n_2) be in $\mathbb{Z} \times \mathbb{N}$, define $(m_1,n_1) \leq (m_2,n_2)$ to be the dictionary ordering $n_1 < n_2$ or both $n_1 = n_2$ and $m_1 \leq m_2$.

Remark 2.1. Definition 2.3 defines a total order on $\mathbb{Z} \times \mathbb{N}$.[7]

Lemma 2.3. *Let U is a set of positions such that $|U_N|$ is finite for all natural numbers, N, then U is well-ordered by the ordering in Definition 2.3.*

Proof. Let V be a subset of U.
Let $n' = \min\{n \in \mathbb{N} : (m,n) \in V\}$. The set $\{m \in \mathbb{Z} : (m,n') \in V\}$ is equal to $U_{n'}$ which is finite.
Let $m' = \min\{m \in \mathbb{Z} : (m,n') \in V\}$.
$(m',n') \in V$, and $\forall (m,n) \in V$, $(m',n') \leq (m,n)$ so U is well-ordered. □

Notation For the remainder of this section, given a set of positions, U, the elements U are assumed to be indexed by the well-ordering induced by Definition 2.3, and $(m_i,n_i) = u_i$ for all $u_i \in U$.

Given a vector space M, a linear function rule $T : M^\infty \to M^\infty$, and a set of positions U, a linear function on M^∞ can be defined that maps a sequence $x \in M^\infty$ to that values of the positions in U for $\Phi_T(x)$.

Definition 2.4. Let M be an 1-dimensional vector space with scalar field F. Let $T : M^\infty \to M^\infty$ be a bijective linear operator with a lower-triangular matrix. Let U be a set of positions such that the cardinality of U_N is finite for all natural numbers, N. The *completion function* for T with U, denoted is defined as follows.

$S_U^T : M^\infty \to M^{|U|}$ where for all $x \in M^\infty$,

$$S_U^T(x) = [(T^{m_i}(x))_{n_i}]_{i=1}^{|U|} \tag{5}$$

The remaining results in this section show how to use completion functions to determine the completeability of sets of positions.

Proposition 2.4. *Let M be an 1-dimensional vector space with scalar field F. Let $T : M^\infty \to M^\infty$ be a bijective linear operator with a lower-triangular matrix. Let U be a set of positions such that the cardinality of U_N is finite for all natural numbers, N.*

(i) U is completable for T if and only if S_U^T is surjective.

(ii) U is uniquely completable for T if and only if S_U^T is bijective.

Proof. Assume U is completable for T. Let $(y_n) \in M^{|U|}$. Define $d : U \to M$ such that $d(u_i) = y_i$ for all $u_i \in U$. There exists $x_1 \in M^\infty$ with chain $c_1 = \Phi_T(x_1)$ such that $c_1(u_i) = d(u_i)$ for all $u_i \in U$. For all $i = 1, \dots, |U|$.

$$\begin{aligned} S_U^T(x_1)_i &= T^{m_i}(x_1)_{n_i} \\ &= c_1(m_i, n_i) \\ &= d(m_i, n_i) \\ &= y_i \end{aligned} \tag{6}$$

Therefore S_U^T is surjective.

Assume S_U^T is surjective. Let $d : U \to M$ be a function. Notice that $(d(u_i))_{i=1}^{|U|} \in M^{|U|}$. There exists $x_2 \in M^\infty$ such that $S_U^T(x_2) = (d(u_i))$. Let $c_2 = \Phi_T(x_2)$. For all $u_i = (m_i, n_i) \in U$,

$$\begin{aligned} c_2(m_i, n_i) &= T^{m_i}(x_2)_{n_i} \\ &= S_U^T(x_2)_i \\ &= d(m_i, n_i) \end{aligned} \tag{7}$$

Therefore, U is completable for T.

Assume U is uniquely completable for T. By the previous result S_U^T is surjective.

Let $x_k \in \ker(S_U^T)$ so $T^{m_i}(x_k)_{n_i} = 0$ for all $i = 1, \dots, |U|$. Also, let $c_k = \Phi_T(x_k)$. Let $d : U \to M$ where $d(u_i) = 0$ for all $u_i \in U$. For all $u_i = (m_i, n_i) \in U$,

$$c_k(m_i, n_i) = T_i^m(x_k)_{n_i} = 0 = d(m_i, n_i) \tag{8}$$

Let $c_0 = \Phi_T(0)$ so $c_0(u_i) = 0$ for all $(u_i) \in U$. For all $u_i = (m_i, n_i) \in U$,

$$c_0(m_i, n_i) = 0 = d(m_i, n_i) \tag{9}$$

Since U is uniquely completable for T, $c_k = c_0$ so $x_k = 0$. Since $\ker(S_U^T) = \{0\}$, S_U^T is injective.

Assume S_U^T is bijective. By previous result U is completable for T.

Let $d : U \to M$ be a function. Let c_3 and c_4 be in $\mathcal{C}_T$ such that $c_3(u_i) = c_4(u_i) = d(u_i)$ for all $u_i \in U$. Let x_3 and x_4 be the seeds of c_3 and c_4 respectively. For all $i = 1, \dots, |U|$,

$$\begin{aligned} S_U^T(x_3)_i &= T^{m_i}(x_3)_{n_i} \\ &= c_3(m_i, n_i) \\ &= c_4(m_i, n_i) \\ &= T^{m_i}(x_4)_{n_i} \\ &= S_U^T(x_4)_i \end{aligned} \tag{10}$$

Therefore, $S_U^T(x_3) = S_U^T(x_4)$. Since S_U^T is bijective, $x_3 = x_4$, and $c_3 = \Phi_T(x_3) = \Phi_T(x_4) = c_4$. Therefore, U is uniquely completable for T. □

Solving completion problems utilizes many properties of infinite matrices that can be referenced in Cooke [4]. The next proposition addresses sets of positions for which a completion function is not defined.

Proposition 2.5. *Let M be an 1-dimensional vector space with scalar field F. Let U be a set of positions such that for a natural number, N, $N < |U_N|$, then U is not completable for any bijective linear operator $T : M^\infty \to M^\infty$ with a lower-triangular matrix.*

Proof. If $|U_N|$ is finite, then S_U^T is defined. Let $m_\circ \in M$ where $m_\circ \neq 0$.

Let $b_k = (0, \dots, 0, m_\circ, 0, \dots)$ where $m_\circ$ is in the kth position for all $k \in \mathbb{N}$. b_k is clearly a basis for M^∞.

For $k > N$, $S_U^T(b_k) = (T^{m_i}(b_k)_{n_i})_{i=1}^{N+1} = 0$ since the first N terms of b_k are 0. Therefore, $\mathrm{Im} S_{U_N}^T \subseteq \mathrm{span}\{S_{U_N}^T(b_1), \dots, S_{U_N}^T(b_N)\}$.

$$\dim \mathrm{Im} S_{U_N}^T \leq \dim \mathrm{span}\{S_{U_N}^T(b_1), \dots, S_{U_N}^T(b_N)\} \leq N < |U_N| = \dim M^{|U_N|} \tag{11}$$

$S_{U_N}^T$ is not surjective so U_N is not completable for T by Proposition 2.4.

If $|U_N|$ is infinite, then U_N contains a finite subset, $\tilde{U_N}$, such that $N < |\tilde{U_N}|$. $\tilde{U_N}$ is not completable by the argument above so U_N is not completable by Lemma 2.1.

Since $U_N \subseteq U$, U is not completable for T by Lemma 2.1. □

Proposition 2.6. *Let M be an 1-dimensional vector space with scalar field F. Let U be a set of positions such that for all natural numbers, N, $|U_N| = N$, then U is uniquely completable for any bijective linear operator $T : M^\infty \to M^\infty$ with a lower-triangular matrix.*

Proof. $n_i = i$ for all natural numbers, i. T^m is a bijective linear operator with lower-triangular matrix for all $m \in \mathbb{Z}$ so the matrix of T^m has nonzero entries on the diagonal for all $m \in \mathbb{Z}$. The ith row of the matrix of S_U^T is the ith row of the matrix of T^{m_i}. Therefore, S_U^T has a lower-triangular matrix with nonzero entries on the diagonal so S_U^T is bijective and is uniquely completable by Proposition 2.4. □

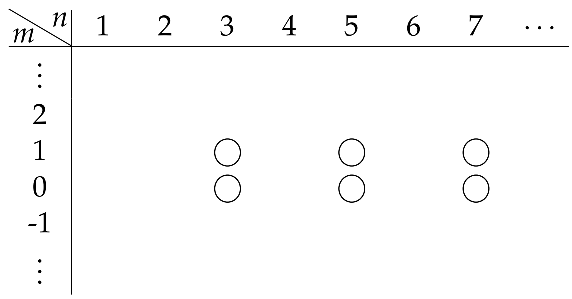

Figure 3.1. Example 3.1

3. Examples of Determining Completeability

Remark 3.1. Let S, T be linear transformations from vector space V to vector space W, and let $P : W \to W$ be a bijective linear operator such that $S = PV$.

$$\dim \operatorname{Im}(S) = \dim \operatorname{Im}(T) \tag{12}$$

and S is bijective if and only if T is bijective.[4, 8]

Example 3.1. Let $T : \mathbb{R}^\infty \to \mathbb{R}^\infty$ where for $(x_n) \in \mathbb{R}^\infty$, $T(x_n) = (y_n)$ defined as follows:

$$y_n = \begin{cases} x_1 & \text{if } n = 1 \\ x_3 - x_1 & \text{if } n = 3 \\ x_n & \text{if } n \text{ is even} \\ x_n - x_{n-3} & \text{if } n > 4 \text{ and } n \text{ is odd} \end{cases}$$

Let $U = \{(0,n),(1,n) : n \in \mathbb{N}, n > 2, \text{and } n \text{ is odd}\}$.

$$\mathcal{M}(S_U^T) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{13}$$

Let A be the matrix row-equivalent to $\mathcal{M}(S_U^T)$ defined as follows.

$$
\begin{aligned}
A &= \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \\
&= Id \\
&= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}
\end{aligned} \tag{14}
$$

S_U^T has an inverse so S_U^T is bijective. Therefore, by Proposition 2.4 is uniquely completable for T.

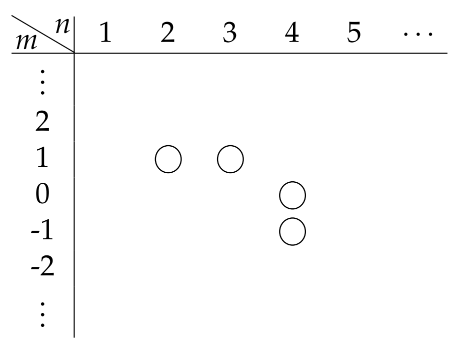

Figure 3.2. Example 3.2

Example 3.2. Let T be the summation operator on $\mathbb{R}^\infty$ with a set of positions $U = \{(1,2),(1,3),(-1,4),(0,4)\}$.

$$
\mathcal{M}(S_U^T) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix} \tag{15}
$$

Let A be the matrix row-equivalent to $\mathcal{M}(S_U^T)$ defined as follows.

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}
\end{aligned} \tag{16}
$$

$\dim \mathrm{Im}(S_U^T) = \mathrm{rank}(A) = 3 < 4$ so S_U^T is not surjective. Therefore, U is not completable for T by Proposition 2.4.

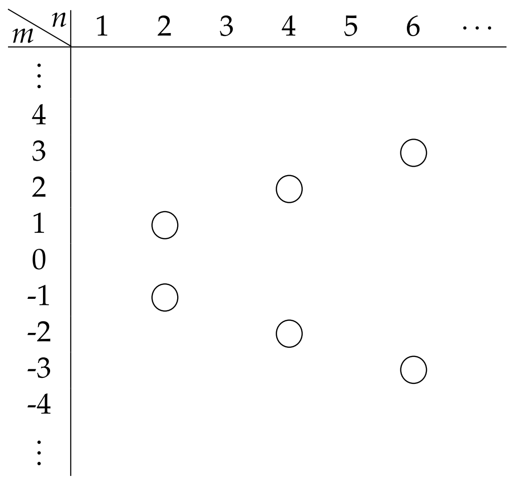

Figure 3.3. Example 3.3

Example 3.3. Let T be the summation operator on $\mathbb{R}^\infty$ with a set of positions $U = \{(n,2n),(-n,2n) : n \in \mathbb{N}\}$.

$$\mathcal{M}(S_U^T) = \left(\begin{array}{cccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 2 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 21 & 15 & 10 & 6 & 3 & 1 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & -1 & 3 & -3 & 1 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ & & & & & & & \cdots & n & 1 & \cdots \\ & & & & & & & \cdots & -n & 1 & \cdots \\ & & & & & & & & \vdots & \vdots & \end{array}\right) \begin{array}{l} \\ \\ \\ \\ \\ \\ \\ \leftarrow \text{row } 2n-1 \\ \leftarrow \text{row } 2n \\ \\ \end{array} \tag{17}$$

Let A be the matrix row-equivalent to $\mathcal{M}(S_U^T)$ defined as follows.

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \mathcal{M}(S_U^T)$$
$$= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 4 & 1 & 4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 2 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 21 & 15 & 11 & 3 & 6 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & -1 & 3 & -3 & 1 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \\ & & & & & & & \cdots & 2n & 0 & \cdots \\ & & & & & & & \cdots & -n & 1 & \cdots \\ & & & & & & & & \vdots & \vdots & \end{pmatrix} \tag{18}$$

Since A is lower-triangular with no zeros on the diagonal, A is the matrix of a bijective linear operator. Since A is equivalent to $\mathcal{M}(S_U^T)$, S_U^T is bijective. Therefore, U is uniquely completable for T by Proposition 2.4.

4. Conclusion

(Editor's Note): This installment examined sequence chains from linear functions with the goal of looking for the minimum amount of information necessary to uniquely define a summation chain. The set of positions was defined, and the notion of uniquely completable is based on this definition. Several elegant theorems gave the conditions under which the set of positions is uniquely completable. The fourth and final installment will discuss the notion of convergence for complex-valued summation chains.

References

[1] Apostol, T., 1974, *Mathematical Analysis 2nd Edition*, Addison-Wesley Publishing Company

[2] Brin, M. and Stuck, G., 2002, *Introduction to Dynamical Systems*, Cambridge University Press

[3] Cameron, P. J., 1992, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press

[4] Cook, R. G., 2014, *Infinite Matrices and Sequence Spaces*, Dover Publications, Inc.

[5] Johnson, J. (2017)*Summation Chains of Sequences Part 1: Introduction, Generation, and Key Definitions* AACTO, Vol. 2, Issue 2. https://www.theresearchcortex.com/summationchainsofseqpart1

[6] Johnson, J. (2017)*Summation Chains of Sequences Part 2: IRelationships Between Sequences via Summation Chains* AACTO, Vol. 2, Issue 2. https://www.theresearchcortex.com/summation-chains-of-sequence-p2

[7] Lee, J., 2011, *Introduction to Topological Manifolds*, Springer Science+Business Media LLC

[8] Leon, S. J., 2002, *Linear Algebra with Applications 6th Edition*, Prentice Hall, Inc.

[9] Scheinerman, E. R., 1996, *Invitation to Dynamical Systems*, Dover Publications, Inc.